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Note

On the Zeroes of a Polynomial

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Let $P(z)=\sum_{j=0}^{n-1} a_j z^j + z^n$ $(n \geqslant 2)$ be a polynomial with complex coefficients, where not all of the numbers $a_0, ..., a_{n-2}$ are equal to 0. We prove that if $P(z)=0$, then

$$|z+\tfrac{1}{2}a_{n-1}| \leqslant \tfrac{1}{2}|a_{n-1}| + \left(\sum_{j=2}^{n} |a_{n-j}|\,\alpha^{j-2}\right)^{1/2},$$

with $\alpha = 1/\max_{2 \leqslant j \leqslant n} |a_{n-j}|^{1/j}$.

In this note we designate by P the polynomial $P(z)=\sum_{j=0}^{n-1} a_j z^j + z^n$ $(n \geqslant 2)$, where $a_0, ..., a_{n-1}$ and z are complex numbers. Moreover, we assume that $a_0, ..., a_{n-2}$ are not all equal to 0. The following proposition concerning the location of the zeroes of P was proved by Walsh [4] in 1924.

THEOREM A. *If $P(z)=0$, then*

$$|z+\tfrac{1}{2}a_{n-1}| \leqslant \tfrac{1}{2}|a_{n-1}| + \sum_{j=2}^{n} |a_{n-j}|^{1/j}. \tag{1}$$

New proofs of Theorem A were given by Rudnicki [3] and Bell [1]. In 1970 Rahman [2] presented an interesting refinement of inequality (1).

THEOREM B. *If $P(z)=0$, then*

$$|z+\tfrac{1}{2}a_{n-1}| \leqslant \tfrac{1}{2}|a_{n-1}| + cM, \tag{2}$$

with $M=\sum_{j=2}^{n} |a_{n-j}|^{1/j}$ and $c=\max_{2 \leqslant j \leqslant n}(M^{-1}|a_{n-j}|^{1/j})^{(j-1)/j}$.

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Since $0 < c \leqslant 1$ it follows that (2) sharpens inequality (1). In this note we prove a counterpart of Rahman's result which provides a new refinement of (1).

THEOREM. *If* $P(z) = 0$, *then*

$$|z + \tfrac{1}{2} a_{n-1}| \leqslant \tfrac{1}{2} |a_{n-1}| + \left(\sum_{j=2}^{n} |a_{n-j}| \, \alpha^{j-2} \right)^{1/2}, \tag{3}$$

with $\alpha = 1/\max_{2 \leqslant j \leqslant n} |a_{n-j}|^{1/j}$.

Proof. Let $K = (\sum_{j=2}^{n} |a_{n-j}| \, \alpha^{j-2})^{1/2}$. We assume (for a contradiction) that z does not satisfy inequality (3). Then we obtain

$$|z| + \tfrac{1}{2} |a_{n-1}| \geqslant |z + \tfrac{1}{2} a_{n-1}| > \tfrac{1}{2} |a_{n-1}| + K, \tag{4}$$

which implies

$$|z| > K. \tag{5}$$

Let

$$b_j = |a_{n-j}| \, \alpha^{j-2} K^{-1} \qquad (2 \leqslant j \leqslant n).$$

Since

$$\alpha K = \left(\sum_{j=2}^{n} |a_{n-j}| \, \alpha^{j} \right)^{1/2} \geqslant 1,$$

we get for $j = 2, ..., n$:

$$b_j K^{j-1} = |a_{n-j}| \, (\alpha K)^{j-2} \geqslant |a_{n-j}|. \tag{6}$$

From (5) and (6) we obtain for $j = 2, ..., n$:

$$|a_{n-j}| \leqslant b_j \, |z|^{j-1}$$

and

$$|a_{n-j}| \, |z|^{n-j} \leqslant b_j \, |z|^{n-1}.$$

Summing leads to

$$\sum_{j=2}^{n} |a_{n-j}| \, |z|^{n-j} \leqslant \sum_{j=2}^{n} b_j \, |z|^{n-1} = K \, |z|^{n-1}. \tag{7}$$

From (4) and (7) we get

$$\begin{aligned} |z^n + \tfrac{1}{2}a_{n-1}z^{n-1}| &> (\tfrac{1}{2}|a_{n-1}| + K)\,|z|^{n-1} \\ &\geqslant \tfrac{1}{2}|a_{n-1}|\,|z|^{n-1} + \sum_{j=2}^{n} |a_{n-j}|\,|z|^{n-j} \\ &\geqslant \left|\tfrac{1}{2}a_{n-1}z^{n-1} + \sum_{j=2}^{n} a_{n-j}z^{n-j}\right|. \end{aligned}$$

Hence, we obtain

$$|P(z)| \geqslant |z^n + \tfrac{1}{2}a_{n-1}z^{n-1}| - \left|\tfrac{1}{2}a_{n-1}z^{n-1} + \sum_{j=2}^{n} a_{n-j}z^{n-j}\right| > 0,$$

which contradicts the assumption that z is a zero of P. ∎

Remark. Since

$$\left(\sum_{j=2}^{n} |a_{n-j}|\,\alpha^{j-2}\right)^{1/2} = \sum_{j=2}^{n} |a_{n-j}|\,\alpha^{j-1}\left(\sum_{j=2}^{n} |a_{n-j}|\,\alpha^{j}\right)^{-1/2},$$

with $|a_{n-j}|\,\alpha^{j-1} \leqslant |a_{n-j}|^{1/j}$ $(2 \leqslant j \leqslant n)$ and $(\sum_{j=2}^{n} |a_{n-j}|\,\alpha^j)^{-1/2} \leqslant 1$, we conclude that (3) refines inequality (1). It is natural to ask whether the two bounds given in (2) and (3) can also be compared. The answer is "no." For instance, let

$$P(z) = z^n + a_1 z + a_0$$

with $|a_1|^{1/(n-1)} = |a_0|^{1/n} > 0$ and $n \geqslant 3$. Simple calculations reveal that the bound given by (2) is better than the one given by (3). However, if $\alpha = |a_{n-2}|^{-1/2}$, then (3) provides a better bound than (2). We prove this assertion:

Since $|a_{n-j}|^{1/j} \leqslant \alpha^{-1}$ for $2 \leqslant j \leqslant n$, we get

$$|a_{n-j}|\,\alpha^{j-2} = |a_{n-j}|\,|a_{n-2}|^{-(j-2)/2} \leqslant |a_{n-j}|^{1/j}\,|a_{n-2}|^{1/2}. \tag{8}$$

And, from the Cauchy–Schwarz inequality and (8) we obtain

$$\begin{aligned} \sum_{j=2}^{n} |a_{n-j}|\,\alpha^{j-2} &= \left(\sum_{j=2}^{n} |a_{n-j}|\,\alpha^{j-1}\right)^2 \Big/ \sum_{j=2}^{n} |a_{n-j}|\,\alpha^{j} \\ &\leqslant \sum_{j=2}^{n} |a_{n-j}|\,\alpha^{j-2} \end{aligned}$$

$$\leqslant \sum_{j=2}^{n} |a_{n-j}|^{1/j} |a_{n-2}|^{1/2}$$

$$\leqslant \max_{2 \leqslant k \leqslant n} \left[\sum_{j=2}^{n} |a_{n-j}|^{1/j} |a_{n-k}|^{(k-1)/k} \right]^{2/k}$$

$$= (cM)^2,$$

with c and M as defined in Theorem B.

References

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